The Effect of Exposure Duration on Visual Character Identification in Single, Whole, and Partial Report

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The psychometric function of single-letter identification is typically described as a function of stimulus intensity. However, the effect of stimulus exposure duration on letter identification remains poorly described. This is surprising because the effect of exposure duration has played a central role in modeling performance in whole and partial report (Shibuya & Bundesen, 1988). Therefore, we experimentally investigated visual letter identification as a function of exposure duration. We compared the exponential, the gamma, and the Weibull psychometric functions, all with a temporal offset included, as well as the ex-Gaussian, the log-logistic, and finally the squared-logistic, which is a psychometric function that to our knowledge has not been described before. The log-logistic and the squared-logistic psychometric function fit well to experimental data. Also, we conducted an experiment to test the ability of the psychometric functions to fit single-letter identification data, at different stimulus contrast levels; also here the same psychometric functions prevailed. Finally, after insertion into Bundesen’s Theory of Visual Attention (Bundesen, 1990), the same psychometric functions enable closer fits to data from a previous whole and partial report experiment.

**Keywords:** psychometric function, character identification, Theory of Visual Attention, visual short-term memory, exposure duration

A visual scene typically contains several objects, one or more which are of special importance for us to identify, and some that are not. Being able to quantify and model the accuracy of visual object identification in multi-object displays is important for such diverse areas as reading speed and learning (Pelli & Tillman, 2007; Rasinski, 2000), traffic safety (Baldock, Mathias, McLean, & Berndt, 2007; Richardson & Marotoli, 2003), human–computer interaction (Chen & Chien, 2005; Chien, Chen, & Wei, 2008), and diagnostics of perceptive and cognitive disorders (Behrmann, Nelson, & Sekuler, 1998; Cheong, Legge, Lawrence, Cheung, & Ruff, 2007; Habekost & Starrfelt, 2008).

Whole and partial report experiments concern visual identification in multi-element displays, which can be thought of as simplified visual scenes compared with the more complex ones that we typically encounter in real life. In whole report, a number of simultaneously presented target elements are to be reported by the subject. Partial report is similar to whole report, except that distractor elements (elements that are not to be reported) are concurrently included in the display.

Bundesen’s Theory of Visual Attention (TVA; Bundesen, 1990) offers a quantitative model linking perception of single isolated objects to perception of multiple objects in whole and partial report experiments. The theory assumes that the total amount of perceptual processing resources, which determines the rate of perceptual processing, is limited and independent of the number of target and distractor objects displayed. Processing resources are allocated to both target as well as distractor objects. Through attentional filtering a proportionally smaller amount of processing resources is allocated to distractor objects. After processing resources are allocated, all objects participate in a race for being encoded into the visual short-term memory (VSTM), which has a limited storage capacity. In Appendix A, we present relevant formulas from TVA. The time course of encoding has been very central to TVA, thus the theory has mainly been applied to experiments in which stimulus exposure duration was the independent variable.

When applied to single objects, TVA is reduced to the psychometric function for object identification as a function of stimulus exposure duration but, surprisingly, few studies (for an exception, see Bundesen & Harms, 1999) have studied this topic. This is our motivation for studying the psychometric function for letter identification as a function of stimulus exposure duration in more detail. We briefly describe the six psychometric functions we included in the study. In Appendix B, we give further details.

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The exponential distribution with a temporal offset included was used as the psychometric function in TVA (Bundesen, 1990). Using this psychometric function rests on the assumption that encoding into VSTM can be considered events from a homogenous Poisson process, for which the waiting times are exponentially distributed. From a psychophysical perspective, we find it strange that there should be a fixed temporal offset with no variation between trials before encoding can take place, which is what this psychometric function implies. Given the variability typically observed in psychophysical experiments, we find a discontinuous psychometric function unlikely. Instead, we find it more plausible that the inherent variability of the brain will smear the psychometric function so that the average encoding rate rises as a smooth function of exposure duration much as Dyrholm, Kyllingsbæk, Espeseth, and Bundesen (2011) found. The gamma, the Weibull, and the ex-Gaussian distributions all represent generalizations of the exponential distribution, and all of these are smooth functions.

The Weibull distribution has previously been used for modeling the psychometric function of visual contrast detection, visual discrimination, as well as visual identification when these were investigated as a function of stimulus contrast (Pelli, 1985, 1987; Pelli, Burns, Farell, & Moore-Page, 2006).

The gamma distribution corresponds to the waiting-time distribution, when waiting for several independent and identically distributed events, that each has an exponentially distributed waiting time-distribution. If correct identification depends on the firing of several independent neural units firing as Poisson processes, then the gamma distribution could describe the psychometric function of identification as a function of stimulus duration. Based on a similar argument the gamma distribution has been fitted to response time distributions (Luce, 1991; Van Breukelen, 1995).

The ex-Gaussian distribution characterizes the sum of an exponential distributed variable and a Gaussian distributed variable. Thus, if Gaussian noise is added to the temporal offset in the exponential distribution the waiting times for perceptual processing would be distributed according to the ex-Gaussian distribution. The ex-Gaussian has been used for modeling reaction time (RT) data (Luce, 1991), and Dyrholm et al. (2011) used it in the TVA framework to model performance in whole and partial report tasks.

An important question is what causes the shape of a psychometric function? Clearly the shape must reflect the construct and limitations of the physical mechanism underlying perception, that is, it must reflect the neural activity in the task relevant areas of the brain. It has previously been demonstrated that individual sensory neurons show response functions (firing rate vs. stimulus intensity) that closely resemble the psychometric function seen in detection tasks, such as the logistic distribution (Lansky, Pokora, & Rospars, 2007). In NTV, which is a neural interpretation of TVA (Bundesen, Habekost, & Kyllingsbæk, 2005), it is assumed that the rate at which stimuli are perceptually processed is proportional to neural firing rates in the visual cortex. Mathematically, the processing rate is the hazard rate. Further descriptions of the concept of hazard rate can be found in several texts (Aalen & Gjessing, 2001; Luce, 1991; Van Zandt, 2002). The processing rate is explicit in the exponential function where it equals the parameter, \( \lambda \). For other psychometric functions, the hazard rate is generally not explicit but can easily be derived (Appendix A).

Exposing cats and monkeys to transient stationary gratings with a duration of 200 ms, Albrecht, Geisler, Frazor, and Crane (2002) mapped out the instantaneous firing rates of responsive neurons in the visual striate cortex. The typical temporal profile of the firing rates is similar to the profile that Bundesen and Habekost (2008, p. 116) expected to follow the abrupt onset of a stimulus: “When a stimulus appears abruptly (a kind of successive contrast), firing rates of typical neurons responding to the stimulus first increase rapidly, then reach a maximum, and finally decline and approach a somewhat lower, steady state level.” Therefore, it is likely that the psychometric function for object identification as a function of exposure duration has a nonmonotonic hazard rate. Accordingly, a preliminary report (Shibuya & Bundesen, 1994) described the hazard function in a 2-AFC discrimination task, in which exposure duration was varied, as having a nonmonotonic hazard rate. However, all of the functions described above have monotonic hazard rates. Therefore, we find it worthwhile to consider also two psychometric functions that have nonmonotonic hazard functions.

The log-logistic is such a distribution, because with appropriately chosen parameters; it has a unimodal, and hence nonmonotonic hazard function. The log-logistic (or Fisk) distribution is the probability distribution of a random variable whose logarithm has a logistic distribution. It has been used for modeling various kinds of diffusion processes (Brüderl & Diekmann, 1995; Diekmann, 1992). Also, it has been used for modeling proportion correct in single-digit identification as a function of contrast (Strasburger, 2001).

The squared-logistic distribution is another distribution with a nonmonotonic hazard function. We describe the squared-logistic because we found it to be a simple function which has a hazard function that closely resembles the instantaneous firing rate of single neurons in the visual cortex like those depicted by Albrecht et al. (2002). Compared with the hazard function of the log-logistic distribution the hazard function of the squared-logistic distribution seems to drop off faster after the peak, and furthermore the hazard approaches a quasi-stationary level rather than continuing to drop off as exposure time increases.

To find the most appropriate psychometric function, we evaluated each of the six functions described earlier on four data sets, two of which are from original experiments and two of which stem from previous experiments.

In Experiment 1, we investigated the psychometric function of single-letter identification as a function of exposure duration. In some TVA-based studies (Bundesen & Harms, 1999; Shibuya & Bundesen, 1988), performance is averaged across the different letter identities. Therefore, as an initial approach, we average performance across the different letter identities although there is no a priori reason to assume that the psychometric function preserves its shape when averaged across several stimuli in an identification task. In fact, the psychometric function for identification of individual letters as a function of contrast was investigated by Alexander, Xie, and Derlacki (1997). This study demonstrated that the psychometric function for identification of 10 Sloan letters depended significantly on letter identity. Also, a recent study by Kyllingsbæk, Markussen, and Bundesen (2011) showed that TVA parameters differ for different digits and for Landolt rings of different orientation. Hence, we find that it is reasonable to assume that averaging over letter identities may affect the shape of the psychometric function. Therefore, we also fit the 6 psychometric functions to the data without averaging across letter identity.
Experiment 2 is similar to Experiment 1 but the contrast is varied between 11 different levels; the number of repetitions is lowered and shorter exposure durations are used for higher stimulus contrast levels. Bundesen and Harms (1999) showed that the psychometric function for letter identification as a function of exposure duration is very steep for high contrast stimuli. This is problematic because it is difficult to present letter stimuli with high enough temporal resolution to obtain an accurate characterization of the dynamic part of the psychometric function. Therefore, in Experiment 1 we decided to use a lower contrast level than that used by Bundesen and Harms (1999). The effect of changing the stimulus contrast level was demonstrated by Di Lollo, von Mühlenen, Enns, and Bridgeman (2004; Figure 4) when they investigated visual single-object identification, also as a function of exposure duration. In their study, proportion correct as a function of exposure duration was plotted for four different contrast levels, and from visual inspection, it is evident that identification accuracy drops as stimulus contrast is decreased. However, they did not fit psychometric functions to their data. We find it is reasonable to ask whether different stimulus contrast levels might also favor different types of psychometric functions for single-letter identification as a function of exposure duration, or alternatively that the same psychometric function can be used, only with different parameter values. The question is particularly relevant here because the psychometric function determined in Experiment 1 accounts for identification of stimuli that are displayed at a relatively lower contrast than has traditionally been the case in TVA studies (Bundesen & Harms, 1999; Shibuya & Bundesen, 1988).

Bundesen and Harms (1999) investigated the psychometric function of letter identification as a function of exposure duration. In this study, the exponential psychometric function was used to model the data from the three subjects each carrying out a total of 4000 trials. Bundesen and Harms (1999) did not however, fit any other psychometric function to their data. Here, we compare the fits of the 6 psychometric functions described earlier to their original data.

Shibuya and Bundesen (1988) conducted a whole and partial report experiment, with two observers each completing 6,480 trials. In whole report, observers were presented with 2–6 visual targets (digits). In partial report, up to 8 distractors (letters) were presented with the targets. Observers’ performance was recorded as the proportion of scores of j or more (correctly reported targets). Shibuya and Bundesen showed that TVA could account very well for the observed data. Until now, TVA has assumed an exponential psychometric function but in Appendix A we show how to generalize it to allow for other psychometric functions by letting the encoding process be a nonhomogeneous Poisson process. Here, we insert each of the six psychometric functions described above into TVA and test each of these six models against Shibuya and Bundesen’s original data.

Model comparison is a difficult and complex task and no single method can provide a final solution. On a general level, three factors need to be taken into account. First, the goodness-of-fit describes how well the model describes the data. A good fit is obviously a necessary although not sufficient criterion for accepting a model. Second, model flexibility must be taken into account as more flexible models generally provide a better fit. As model flexibility increases with the number of free parameters several model evaluation criteria such as Akaike’s and Bayesian Information Criteria (AIC; Akaike, 1974 and BIC, Schwarz, 1978), which we employ here, are based on correcting the goodness-of-fit with the number of free parameters. However, two models with the same number of parameters can differ in flexibility, so the number of free parameters does not entirely capture model flexibility. Here, in addition to AIC and BIC, we employ cross-validation (CV, Myung, Pitt, & Kim, 2005), which is not based on the number of free parameters. In CV the model is tested on subsamples of the data not used for fitting the model. This procedure is repeated so that the model is ultimately tested on all the available data. This takes model-flexibility into account because more flexible models have the disadvantage of overfitting; that is, closely matching the sampling variability in the data. Matching the sampling variability in the data used for fitting the model will not make the model match the sampling variability in the data used for testing the model and hence the more flexible model will not generally provide a better fit. The third factor that is an important criterion for model selection is model interpretability. If we can interpret the parameters of the model we can relate it to other findings based on other data sets and in this way increase our understanding of the underlying system. Here we will consider all of these factors in our comparison of the six models.

In summary, our aim is to investigate the psychometric function of visual identification as a function of exposure duration. We evaluate six selected psychometric functions for single letter identification when averaging data over letter identities and also for each individual letter identity. By inserting each psychometric function into TVA we also evaluate their appropriateness for describing performance in whole and partial report.

Method

Paradigm

The task was to report a single stimulus letter cued by a fixation point and terminated by a mask. The report was carried out as a forced choice procedure with 26 alternatives (26-AFC). A trial commenced with the fixation point marker being displayed for 1000 ms. Immediately after this the stimulus letter was shown. A randomized mask, lasting for 500 ms, followed the stimulus and then the report display consisting of the 26 letters of the alphabet was shown. The subject had to report which stimulus letter he thought was the one presented by typing the letter on a standard Danish keyboard. When a letter was pressed the corresponding letter in the alphabet would blink, providing feedback to the subject. After this, the alphabet disappeared and a new trial would start. Each time the subject had carried out 100 new trials a message was displayed on the screen stating how many trials remained. On average, a trial took about 3 s. The stimulus conditions varying between trials were the identity of the letter and the exposure duration. All experimental blocks contained all stimulus conditions twice but the order of the stimulus conditions was randomly permuted for each block.

Subjects

Two Danish students of engineering as well as one of the authors served as subjects. The two students, subject MK and subject MH, were paid by the hour. Subject MK was a 24-year-old...
male with corrected to normal vision (contact lenses), and subject MH was a 21-year-old male, with corrected to normal vision (glasses). Subject AP was a 28-year-old male with normal visual acuity. Subjects MK and MH were naïve about the purpose of the experiment, whereas subject AP was not.

Stimulus Display

The fixation point marker was a dot (·), except for subject AP in Experiment 1, where the fixation point marker was a cross (+). The fixation point marker was shown at the center of the screen, when the system was ready for a trial. The luminance of the point marker was 8.9 Cd/m². The background luminance of the screen was fixed at 45.7 Cd/m² in both Experiment 1 and 2.

The stimulus consisted of a single black/gray capital letter displayed at the fixation point. The letter could be any of the 26 letters of the English alphabet. The font used was New Courier. Examples are shown in Figure 1, A–C. The letter was presented centrally (foveally) at the location of the fixation point. The letter subtended a visual angle of about 1.1° vertically and 1.1° horizontally (foveally) at the location of the fixation point. The letter letters of the English alphabet. The font used was New Courier.

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Mask

The mask consisted of a binary image that was randomly generated for each trial. The procedure that was used for generating the instances of the mask was based on phase scrambling of the stimulus images in the Fourier domain. The procedure is described in detail in Appendix C, and examples are shown in Figure 1, D–F. The mask was shown immediately after the stimulus was removed from the screen. The physical size of the mask covered the area where all possible stimuli from A to Z had been presented, which means that the mask subtended about 1.3° vertically and 1.3° horizontally. The luminance of the binary mask was 45.7 Cd/m² in the light regions and 0.0 Cd/m² in the dark regions.

Report Display

In the report display, the entire alphabet from A to Z was displayed in a single row. The vertical angle between the vertical center of the row of letters displayed and down to the bottom of the screen was 9°. The alphabet was printed in the same size as used for the stimuli letters. The entire row of letters subtended about 1.1° vertically and 33° horizontally. The luminance of the letters in the report display was 32.2 Cd/m².

Apparatus

The subject was seated in front of a computer-driven (NVIDIA GeForce 7950 GT) cathode ray screen (17” Flatron 915FT Plus) at a viewing distance of 57 cm in a dark (0.0 cd/m²) room. The viewing distance was chosen so that 1° of visual angle corre-

sponded to approximately 1 cm on the screen. The refresh rate of the monitor was set to 200 Hz, and the pixel resolution was 480 × 640. The monitor was preheated for at least half an hour before any experimental session was initiated. The experiment was written in Matlab, using the Psychophysics Toolbox (Brainard, 1997; Pelli, 1997).

Analysis

We fitted the psychometric functions and TVA to our data by maximizing the likelihood using a quasi-Newtonian optimization routine provided by the Matlab optimization toolbox. We used a number of random starting points. We also added random noise to the parameters after convergence and then restarted the optimization. This was done to increase the chances of finding a global rather than a local minimum. From the maximum likelihood, we calculated the AIC as 2k – 2ln(L) and the BIC as ln(N)k – 2ln(L), where k is the number of free parameters, N the number of independent data points, and L the likelihood of the model.

In addition, we used 10-fold CV (Myung, Pitt, & Kim, 2005), which is a straightforward yet powerful way to evaluate models. In the 10-fold CV, the original sample is randomly partitioned into 10 subsamples. The partitioning is done so that each subsample contains equally many trials from each of the experimental conditions. Of the 10 subsamples, a single subsample is retained as the validation data for testing the model, and the remaining data is used as training data. The CV process is then repeated 10 times, with each of the 10 subsamples used exactly once as the validation data. Finally, the log likelihoods of the 10 tests are summed to produce the CV model evaluation criterion.

Experiment 1

In this experiment, the negative Weber contrast, (background luminance – stimulus luminance)/background luminance, of the stimulus letters was fixed at 0.0460. The experiment consisted of 65 sessions for each subject. Within a session each letter was presented two times at all exposure durations. There were 16 different exposure durations: 35, 40, 45, 50, 55, 60, 70, 80, 90, 100, 115, 130, 145, 165, 185, and 210 ms. With this setup, a session contained a total of 2 × 26 × 16 = 832 trials. The time to complete a session was about 40 min; after each session the subject took a break for 10 min. All three subjects participated in this experiment. A subject was not allowed to complete more than 5 sessions per day and was told not to engage in any more sessions if they felt tired. Three subjects, each completing 65 sessions, thus results in a total of 3 × 65 × 832 = 162,240 trials.

Experiment 2

In this experiment the stimulus was shown at 11 different contrast levels. Within each session, the stimulus contrast level was fixed while it varied randomly between sessions. A total of 5 sessions contained the same contrast level, and within each session each condition was repeated two times. In total 55 sessions were completed. After each session the subject took a break for 10 min. Only subject AP participated in this experiment. The subject did not complete more than 6 sessions per day and further did not engage in any more sessions if he felt tired.
For the stimuli having the negative Weber contrast levels of 0.083, 0.046, 0.028, 0.020, and 0.018 the exposure durations were the same as the ones in Experiment 1, that is: 35, 40, 45, 50, 55, 60, 70, 80, 90, 100, 115, 130, 145, 165, 185, and 210 ms. The time to complete one of these sessions was about 40 min. This session type contained a total of $\frac{26}{11}$ trials. For the stimuli having the negative Weber contrast levels of 0.370, 0.210 and 0.129, the exposure durations were: 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60, and 65 ms. The time to complete one of these sessions was about 30 min. This session type contained a total of $\frac{26}{12}$ trials. For the stimuli having the negative Weber contrast levels of 1.000, 0.906, and 0.626 the exposure durations were: 0, 5, 10, 15, 20, 25, 30, and 35 ms. The time to complete one of these sessions was about 20 min. This session type contained a total of $\frac{832}{26}$ trials.

One subject completing each type of session five times for each contrast level thus results in a total of $36,400$ trials.

**Results**

To examine which psychometric function would describe the data when averaged across letter identity, we fitted each of the six psychometric functions to the data from Experiment 1. The guessing rate was set to $\gamma = \frac{1}{26}$, and the lapsing rate ($\lambda = 1\%$) was estimated from the data, as the average of the survivor function, $1 - \psi$, (at 165, 185, and 210 ms). In Figure 2, we show the model error (the signed residual) as a function of exposure duration. Residuals are shown for the different psychometric functions and, comparing between the three different subjects, it is seen that the residuals vary systematically with exposure duration; notably, all of the psychometric functions overshoot around 40 to 60 ms and immediately after undershoot slightly less around 60 to 70 ms. Figure 2 shows that the exponential is the least optimal model in the comparison, and furthermore, it is apparent that the residual for the squared-logistic and log-logistic are relatively small compared with the residuals of the other psychometric functions.

To further illustrate how well the psychometric functions fit the data, the proportion correct as a function of exposure duration was averaged over all letter identities is shown in Figure 3. Also shown are the fit of the exponential psychometric function used in TVA and the fit of the squared-logistic psychometric function, which we found provided the best fit (Figure 2). For all subjects (AP, MH, and MK), it is clear that for short exposure durations the proportion correct is slowly rising up until 20% correct, which is not predicted by the exponential psychometric function, which rises abruptly in the beginning. Also for all subjects the exponential function seems to level off too slowly when performance reaches ceiling at longer exposure durations.

To study whether the hazard function of the psychometric function, $F$, resembles the development of firing rates we show the

![Figure 2](image-url)  
*Figure 2.* Residuals, plotted as a function of exposure duration, from Experiment 1. Error bars—too small to be distinguished clearly—show the SEM. There is one graph for each subject: (A) AP, (B) MH, and (C) MK.
hazard functions of the various models in Figure 4 with the empirical hazard rates. The empirical hazard rate was calculated as the linear slope of the cumulative hazard function, which was determined as the negative log of the survivor function. Across all subjects we consistently see that the empirical hazard rate rises smoothly and then falls to some lower level. Note, however, that the uncertainty of the hazard estimate increases as a function of exposure duration. The uncertainty bars (showing SD) in the plot were obtained with the help of bootstrap analysis (Efron & Tibshirani, 1994), which consisted of fitting 100,000 random resamples of the data and calculating the standard deviation of these fits at each experimental condition. Note also that since the hazard is calculated between neighboring data points the exposure durations used in this plot are averages of two consecutive exposure durations. Furthermore, for long exposure durations, as the psychometric function reaches ceiling, the hazard estimates become very imprecise or even infinite (Van Zandt, 2002); therefore, we left these later estimates out of the plots.

To study to what extent the psychometric function depends on letter identity, we also fitted each of the psychometric functions to the data from Experiment 1 without averaging over letter identities. The guessing rates were allowed to vary between letter identities, however the lapsing rates were not; that is, these were the same as described above for the averaged data. To illustrate the parameter differences between letters, in Figure 5, we show parameter histograms as well as parameter scatter plots for the log-logistic psychometric function fitted to the data of subject AP. Under the assumption that the psychometric function obtained by averaging over letter identities is the true psychometric function we applied bootstrapping (Efron & Tibshirani, 1994) to check whether the variance in the parameters of the psychometric functions for the individual letter identities can be ascribed entirely to an effect of random sampling or if it needs also to be ascribed to some systematic effect of letter identity. The bootstrapping consisted of fitting the psychometric function to 200 random resamples of the data and calculating the standard deviation of the estimated model parameters at each experimental condition. The ovals in Figure 5 demark the bootstrap estimated confidence region to which we would expect 95% of the letters, placed according to their individual parameters, to be located. Clearly, many letters are located outside the ovals. This shows that the model parameters vary significantly with letter identity.

To examine how the psychometric function as a function of exposure duration depends on contrast, in Figure 6 we show proportion correct for the various contrast conditions in Experiment 2. For this experiment we note that the guessing rate, $\gamma = 1/26$, and further we shall assume that the lapsing rate, $\lambda = 0$. The fit by the squared-logistic psychometric function is also shown in

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**Figure 3.** Proportion correct in Experiment 1 averaged over letter identities. Error bars show the SEM. The fit of the exponential and the squared-logistic psychometric functions are shown as well. The rectangular insets show $2 \times$ magnification of critical curve sections in which the exponential function systematically first undershoots then overshoots. There is one graph for each subject: (A) AP, (B) MH, and (C) MK.
Figure 6. The fit is very close for all the contrast levels that we investigated.

With the aim of investigating if any of the alternative psychometric functions can improve the ability of TVA to describe whole and partial report data, we inserted each of the six psychometric functions in TVA (Appendix A). The six models were fitted to the data for each of the two subjects of Shibuya and Bundesen (1988). Clearly the squared-logistic fits closer than the exponential psychometric function; this is true for both subject MP (Figure 7A) and subject HV (Figure 7B). The squared-logistic, for example, is able to account for correct reports at very short exposure durations because it does not rise abruptly in the beginning as does the exponential psychometric function used by Shibuya and Bundesen (1988).

Model feasibility for the different experiments is shown in Figure 8. The figures show AIC, BIC, and CV values for each psychometric function applied to each data set. The values are summed over all subjects. There are two graphs for Experiment 1, one for fitting the data averaged over letter identities (Figure 8A) and one for fitting the data not averaging over the individual letter identities (Figure 8B). In both graphs, comparing models using any of the three measures, we see that the exponential is the poorest and the squared-logistic and the log-logistic are the best psychometric functions. The model performance for the data in Experiment 2 is shown in Figure 8C, we see that the ranking of the various models is quite similar to that found for Experiment 1 and that AIC, BIC, and CV seem consistent in their ranking of the models. In Figure 8D we show model performance with respect to modeling Shibuya and Bundesen’s (1988) whole and partial report data. The ranking of the psychometric functions appear similar to what was seen for Experiments 1 and 2, and there appears to be a consistency between AIC, BIC, and CV measures. Finally, Figure 8E shows model performance with respect to modeling Bundesen and Harms’ (1999) data. For this experiment, we again see that both the squared-logistic and the log-logistic are better models than the exponential psychometric function. However we see that the Weibull and gamma psychometric functions fit the data from this experiment better, which might appear curious if we compare this result with the results from modeling the other datasets. However, for most of the trials in these experiments performance
had reached ceiling level. Therefore, because of fewer informative trials, we should be careful about putting too much weight on this dataset in the decision about which psychometric function is the most appropriate one to use.

An overview of model feasibility summed over all experiments, when averaging over letter identities, is shown in Figure 8F. The graph shows the sum of the AIC, BIC, and CV measures for the data in Experiment 1 and 2, as well as the data in Shibuya and Bundesen (1988) and Bundesen and Harms (1999). To ensure consistency with the way the other datasets were modeled, for Experiment 1 we include the AIC, BIC, and CV values obtained after fitting the data when data was averaged over letter identities. Thereby the values found when modeling the individual letters in Experiment 1 (cf. Figure 8B) are not included in the sums shown in Figure 8F. From Figure 8F we see that the three-parameter squared-logistic is the best model, with the two-parameter log-logistic model as the runner-up. The two-parameter exponential distribution is clearly the poorest model. The ranking appears consistent over AIC, BIC, and CV measures. For the most optimal psychometric function for individual letters we refer to Figure 8B, which shows that although the measures disagree whether the squared-logistic or the log-logistic psychometric function provides the best fit the three measures do agree that the squared-logistic and log-logistic provide a much better fit than the exponential psychometric function.

**Discussion**

TVA has been successfully used to model data from whole and partial report type of experiments (Shibuya & Bundesen, 1988). For this, TVA used the exponential distribution as the underlying psychometric function; however, because many other psychometric functions exist, we wondered whether a more optimal psychometric function could be found. In order to answer that question we conducted Experiment 1, which was a single-letter identification experiment, similar to that of Bundesen and Harms (1999), in which exposure duration was varied at a single fixed contrast level. Our first idea was to generalize the psychometric function by simply including an additional, third parameter. The Weibull, the gamma and the ex-Gaussian all represented simple generalizations of the exponential psychometric function, and all of these proved to be better models as measured by AIC, BIC, and CV comparisons. This finding is in good agreement with the findings of...
Dyrholm et al. (2011), who found that the ex-Gaussian provided a better fit to whole and partial report data.

From NTVA (Bundesen et al., 2005) comes the prediction that the hazard rate of the psychometric function should follow the firing rates of neurons in the visual cortex, which have been shown to develop nonmonotonically over time (Albrecht et al., 2002). Therefore, we included two additional psychometric functions, which both have a nonmonotonic hazard function: the log-logistic which is a well-known distribution and the squared-logistic, which is a biologically inspired distribution we developed ourselves. The three-parameter squared-logistic generally produced the best results, but the two-parameter log-logistic came out as the runner-up, comparing all six different psychometric functions (Figure 8).

To illustrate how well the different psychometric functions fit the single-letter identification data from Experiment 1 in Figure 2A–C, we showed the model error as a function of exposure duration for the three different subjects. Comparing across the three subjects, it is clear that the error function develops systematically over time. Even for the best fitting psychometric function, that is the squared-logistic, it appears that first there is an overshoot around 40 to 60 ms and then an undershoot around 60 to 70 ms. In Figure 4, we compared the empirically estimated hazard functions with the hazard functions of the various psychometric functions used. When seen across all subjects in Experiment 1, it is seen (from both Figure 2 and Figure 4) that we find that there is a small, but consistent, systematic misfit as a function of exposure duration, and so we invite future studies to find an even better suited model than we did for characterizing the temporal development of identification accuracy.

The current study contains enough trials per subject to allow us to accurately fit the psychometric function for each individual letter. In the study by Bundesen and Harms (1999) averages over all stimuli letters were used. Our study offers novel knowledge about how dependent the parameters of the psychometric function are on letters identity. Making use of bootstrapping (Efron & Tibshirani, 1994), we found that the parameters of the estimated psychometric functions of the individual letters vary more than would be expected from random sampling variance alone. This is in good agreement with the results of Kyllingsbæk et al. (2011) who used digits and Landolt rings as stimuli and found that the psychometric function depended on the identity of the stimulus. It is, however, not uncommon to average the psychometric function over letter identities, and therefore it is relevant to ask whether it is reasonable to use the same type of psychometric function, both when fitting individual letters, as well as when fitting the data averaged over letters identities. Our results showed that the log-logistic and squared-logistic psychometric functions are optimal in both of these two situations. This means that although the model parameters depend on letter identity, the type of psychometric function does not. Therefore, it still seems reasonable in many types of experiments to average over letter identities to reduce the demand on the total number of trials.

Worried that our search for an optimal psychometric function would suffer from being contrast-specific, we conducted Experiment 2 to verify the performance of the various psychometric functions at a number of different stimulus contrast levels. The result of fitting the data from Experiment 2 is illustrated in Figure 8C. For Experiment 2 the ranking of the psychometric functions was quite similar to the one we had previously seen for Experiment 1. In Figure 6 we showed how well the squared-logistic psychometric function fits the data from Experiment 2.

It is interesting to note that the two best-fitting models we found, namely the log-logistic and the squared-logistic, both have a nonmonotonic hazard function in line with the prediction of Bundesen and Habekost (2008). Other distributions including the Cauchy and the log-normal also have a nonmonotonic hazard function, but these distributions did not provide as good fits (not shown) as the log-logistic and squared-logistic did. The log-logistic distribution
Figure 7 (opposite).
is a special case of the Burr distribution, which again is a special case of the generalized beta distributions of the second kind (Bookstaber & McDonald, 1987). Likewise, the exponent of the squared logistic can be allowed to vary as a free parameter instead of being fixed to the value 2. Little improvement in fitting was found when testing these generalizations (not shown), which is why here we do not go into more detail on these functions. Nonmonotonic hazard functions are described in Aalen and Gjesing (2001).

After insertion of each of the six psychometric functions into TVA, further described in Appendix A, the three-parameter squared-logistic enabled the closest fits (Figure 8D) to the data from (Shibuya & Bundesen, 1988). In (E) we see the results from fitting our models to the data from the whole and partial report experiment in (Shibuya & Bundesen, 1988). In (E) we see the results from fitting our models to the single-letter identification data from (Bundesen & Harms, 1999). Finally, in (F) we see AIC, BIC, and CV measures cumulated over all datasets from the various experiments, including only the fit to the average single-letter data (i.e., not the fit to the individual letters) for Experiment 1.

is a special case of the Burr distribution, which again is a special case of the generalized beta distributions of the second kind (Bookstaber & McDonald, 1987). Likewise, the exponent of the squared logistic can be allowed to vary as a free parameter instead of being fixed to the value 2. Little improvement in fitting was found when testing these generalizations (not shown), which is why here we do not go into more detail on these functions. Nonmonotonic hazard functions are described in Aalen and Gjesing (2001).

After insertion of each of the six psychometric functions into TVA, further described in Appendix A, the three-parameter squared-logistic enabled the closest fits (Figure 8D) to the data from (Shibuya & Bundesen, 1988). A close runner-up was the log-logistic psychometric function.

In general over all datasets that we fitted (single-letter as well as whole and partial report), it is clear (Figure 8) that the squared-logistic is the most optimal psychometric function of the ones that we have considered. An alternative to using the three-parameter squared-logistic psychometric function is to use the log-logistic psychometric function, which comes in as a close runner-up. Despite having only two parameters, the log-logistic still provides very good fits to the data. It is worth noticing that Bundesen’s exponential psychometric function also had two parameters; however the log-logistic provides much better fits.

To this point, we have only taken the goodness-of-fit and model flexibility into account into our assessment. We now turn to the issue of model interpretability. The gamma and ex-Gaussian functions can be seen as generalizations of the exponential function. Hence, two of their parameters map directly onto the two parameters of the exponential. This gives them the advantage of being interpretable in the context of TVA. They also offer simple mechanistic explanations of why the hazard rate vary the way it does.

We can interpret the gamma distribution as the distribution arising from waiting on multiple Poisson processes while the ex-Gaussian...
carries the assumption that Gaussian noise influences the threshold. The Weibull and log-logistic functions are usually not thought of as describing stochastic processes. This is partly because of convention; they can describe stochastic processes although the hazard rate is not explicit in their functional form. However, they lack a mechanistic interpretation of why the hazard rate varies the way it does. This concern is emphasized when seen in the light of TVA, which takes the mechanistic interpretation further when using the race model to explain which objects enter working memory. In other parts of the literature the Weibull and log-logistic psychometric functions are used to describe performance as a function of contrast and little concern is given to finding a mechanistic explanation of the functions’ exact shape (Pelli et al., 2006; Strasburger, 2001). Introducing these functions to the TVA framework could perhaps open for integration with this part of the literature. In order to do this we may need to refer to Bloch’s law, which describe equivalency between stimulus exposure duration and contrast at least for detection of unmasked presentations of less than 100 ms duration (Gorea & Tyler, 1986). If we can draw the parallel to identification of letters followed by a pattern mask, the processing rate of TVA could perhaps be equivalent to contrast sensitivity. More research is needed to clarify whether this path is viable. Here we shall only argue that the introduction of conventional psychometric functions also carry an advantage of being more interpretable in they relate TVA to another large body of literature. Finally, we must admit a weakness of the squared-logistic function. Although it seems superior in terms of goodness-of-fit and model parsimony we can offer no interpretation of its parameters and we cannot relate it to any body of previous work.

Closely linked to the question of model interpretability is the issue of usefulness. The TVA has been used to measure performance of attention and working memory from whole and partial report experiments (Bublak et al., 2005; Finke et al., 2005; Finke et al., 2010). The measure of performance derived from the psychometric function is usually the processing rate, that is, the hazard rate, which can be related to perceptual and attentional ability. Whether the additional degree of freedom in the gamma and in the ex-Gaussian also can be understood qualitatively in terms of human performance remains to be shown. Perhaps the number of trials necessary to estimate three parameters instead of two might make this impractical especially in patient studies. The same problem applies, of course, to the Weibull and squared-logistic functions. Not so for the log-logistic function, which, having the same number of free parameters as the exponential function may have its parameters defined just as precisely as the exponential function fitted to the same data set. A possible added advantage of using the log-logistic function could be that it would allow TVA-based patient studies to be related to patient studies using contrast sensitivity as the measure of performance (Owsley, 2003).

In summary, we investigated visual letter identification as a function of exposure duration and described the squared-logistic, a psychometric function we found no previous accounts for, and which we developed motivated by NTVA (Bundesen et al., 2005) and single-neuron studies by Albrecht et al. (2002). Both the squared-logistic and the well-known log-logistic can model a nonmonotonic hazard function and both of these two psychometric functions fit well to experimental data from single-letter identification experiments; finally inserted into TVA, both psychometric functions improve fits to data from whole and partial report type of experiments.

For all datasets that we modeled we found that the three-parameter squared-logistic and the two-parameter log-logistic were clearly better models than the two-parameter exponential psychometric function, which has until now been used with TVA.

References


EFFECT OF EXPOSURE DURATION


TVA (Bundesen, 1990) provides an account of performance in whole and partial report. In TVA any target or distractor is denoted as an element, and further it is assumed that a subject only correctly identifies the elements that are stored in visual short-term memory. Further, subjects only obtain a chance to store an element if the element is encoded.

Encoding

The hazard rate that a particular element, $i$, is encoded into VSTM is proportional to how large a portion of processing resources the element receives. Any element in the visual field $S$ receives a certain portion $v_i$ of the total processing capacity $C$, which is assumed to be invariant with respect to the number of elements in the display:

$$ C = \sum_{i \in S} v_i $$

It serves as a simplification to assume homogeneity of the visual display (Shibuya & Bundesen, 1988). This appears a reasonable assumption as long as all elements have the same size, the same contrast, the same eccentricity etc. The homogeneity assumption means that all targets receive the same amount of processing resources denoted $v_T$. In the same way all distractors receive the same amount $v_D$ of processing resources which is proportional to the amount that targets receive. The ratio of processing resources $\alpha$ is defined as:

$$ \alpha = \frac{v_D}{v_T} $$

Let us assume a homogenous display that contains $T$ targets and $D$ distractors. The processing resources $v_i$ of any target is then given by:

$$ C = \sum_{i \in S} v_i \Rightarrow C = TV_i + Dv_D = TV_i + \alpha Dv_i = (T + \alpha D)v_i $$

$$ \Rightarrow v_i = \frac{C}{T + \alpha D} $$

Storage

As we have already seen, TVA describes the factors that determine the probability that any given target in a multi-element visual display is encoded into visual short-term memory, however as VSTM typically has only about 3–4 storage places, TVA assumes that not all elements that become encoded are actually stored. Bundesen (1990) assumes that the occupation of places occurs through a so-called race, that is, any newly encoded element will lead to immediate occupation of one storage place if and only if there is still any storage place left in the VSTM.

Let us define that $f$ and $F$ are the probability density and the distribution function for target encoding respectively. Similarly we also define that $g$ and $G$ are the probability density and the distribution function for distractor encoding.

According to the fixed-capacity independent race model (FIRM, Shibuya & Bundesen, 1988) the probability of a score of $j$ (targets reported) from a display containing $T$ targets and $D$ distractors exposed for $t_e$ seconds can be written as:

$$ P(j; T, D, t_e) = P_1 + P_2 + P_3 $$

where $P_1$ is the probability that the score equals $j$ and the total number of elements (targets and distractors) entering VSTM is less than $K$. The number of distractors entering VSTM is denoted by $m$ and $m \leq \min(D, K-j-1)$. If $j = K$, $P_1 = 0$; otherwise:

$$ P_1 = \left( j \right) \frac{(T)}{(F(t))^{j-1}[1 - F(t)]^{T-j} \times \sum_{m=0}^{\min(D, K-j-1)} \binom{D}{m} [G(t)]^m[1 - G(t)]^{D-m} $$

and $P_2$ is the probability that the score equals $j$ and the total number of elements equals $K$ and the $K^{th}$ element entering the VSTM is a target. The number of distractors entering VSTM denoted by $m$ is always $K - j$. If $j = 0$, or $j < K - D$, $P_2 = 0$; otherwise:

$$ P_2 = \left( j - 1 \right) \frac{(T-1)}{(j-1)} \frac{(j)}{(F(t))^{j-1}[1 - F(t)]^{T-j} \times \left( \frac{(D)}{m} \right) [G(t)]^m[1 - G(t)]^{D-m} \int \left( \frac{T}{t} \right) f(t) dt $$

and $P_3$ is the probability that the score equals $j$ and the total number of elements equals $K$ and the $K^{th}$ element entering the VSTM is a distractor. The number of distractors entering VSTM denoted by $m$ is always $K - j$. If $j = K$, or $j < K - D$, $P_3 = 0$; otherwise:

$$ P_3 = \left( j \right) \frac{(T)}{(F(t))^{j-1}[1 - F(t)]^{T-j} \times \left( \frac{(D-1)}{m-1} \right) [G(t)]^{m-1}[1 - G(t)]^{D-m} \int \left( \frac{D}{1} \right) g(t) dt $$

(Appendices continue)
Shibuya and Bundesen (1988) derived explicit score probabilities under the assumption that encoding proceeds as a homogenous Poisson process. This corresponds to assuming (ignoring the temporal offset) that the hazard rates are constant over time. Our contribution is to allow the hazard rates to be time-varying, although we still assume that the hazard rates (for the different elements presented) are mutually proportional functions of time (Bundesen, 1990, 1993, 1998). The way we relax this assumption is explained in the following where we derive a set of generalized FIRM equations. For the generalized FIRM equations, which are characterized by a non-homogenous Poisson process for encoding, with hazard $\lambda_i(t)$ and cumulative hazard function $\Lambda_i(t)$, we can write the following probability distribution function for target encoding:

$$F(t) = 1 - \exp(-\Lambda_i(t))$$

By differentiating the probability distribution using the chain rule we arrive at the probability density function for target encoding:

$$f(t) = \exp(-\Lambda_i(t))\lambda_i(t)$$

Similarly, for distractors we can formulate the probability density function $g$ and the probability distribution function $G$ in terms of the hazard $\lambda_d(t)$ and the cumulative hazard function $\Lambda_d(t)$ for distractor encoding.

We are now able to derive a set of explicit score probabilities that are valid under the generalized FIRM conditions. By inserting the expressions for $F(t)$ and $G(t)$ it is straightforward to calculate the probability $P_j$ when the cumulative hazard functions $\Lambda_i(t)$ and $\Lambda_d(t)$ are known:

$$P_j = \left(\frac{T}{j - 1}\right)[1 - \exp(-\Lambda_i(t))] [\exp(-\Lambda_d(t))]^{j - i} \times$$

$$\sum_{m=0}^{\min(D, j - 1)} \left(\frac{m}{D}\right)[1 - \exp(-\Lambda_d(t))]^{-m} [\exp(-\Lambda_d(t))]^{D - m}$$

Deriving the expression for $P_2$ is a little more complex:

$$P_2 = \int_0^{T - 1} [F(t)]^{j - 1 - 1} [1 - F(t)]^{j - i} \times$$

$$\frac{D}{m} \left[\frac{G(t)}{1} \right] [1 - G(t)]^{D - m} \left(\frac{T}{j - 1}\right)f(t) dt$$

$$= \left(\frac{T}{j - 1}\right)\left(\frac{m}{D}\right) \int_0^{T - 1} [1 - \exp(-\Lambda_i(t))]^{-1} \exp(-\Lambda_i(t)) \times$$

$$\times [1 - \exp(-\Lambda_d(t))]^{D - m} \exp(-\Lambda_d(t)) \lambda_i(t) dt$$

$$= \left(\frac{T}{j - 1}\right)\left(\frac{m}{D}\right) \int_0^{T - 1} \exp(-\Lambda_i(t)[T - j + 1 + \alpha(D - m)]) \times$$

$$\left[1 - \exp(-\Lambda_i(t))\right]^{j - 1} \left[1 - \exp(-\Lambda_d(t))\right]^{D - m} \lambda_i(t) dt$$

$$= \left\{T - 1\right\}_{j - 1} \left(\frac{m}{D}\right) \int_0^{T - 1} \exp(-\Lambda_i(t)[T - j + 1 + \alpha(D - m)]) \times$$

$$\sum_{a=0}^{\min(D, j - 1)} \left(\frac{m}{D}\right) \sum_{b=0}^{D - m} \left(\frac{m}{b}\right) \left(-1\right)^{a + b} \left[\exp(-\Lambda_i(t))\right] \lambda_i(t) dt$$

$$= \left(\frac{T}{j - 1}\right)\left(\frac{m}{D}\right) \times$$

$$\sum_{a=0}^{\min(D, j - 1)} \left(\frac{m}{D}\right) \sum_{b=0}^{D - m} \left(\frac{m}{b}\right) \left(-1\right)^{a + b} \frac{1 - \exp(-\Lambda_d(t))}{Q}$$

where, in the last two expressions we have substituted

$$Q = T - j + a + 1 + \alpha(D - m + b)$$

In the first step, we insert the expressions for $F(t)$, $G(t)$ and $f(t)$. The second step is a simple reduction using the assumption in TVA that the hazard function for distractors is proportional to the hazard function for targets so that $\lambda_d(t) = \alpha \lambda_i(t)$ and $\Lambda_d(t) = \alpha \Lambda_i(t)$. The third step uses the binomial expansion while the fourth step is again a simple reduction. The fifth step uses integration by substitution, noticing that $\partial \Lambda_i(t)/\partial t = \lambda$, to arrive at the final expression. Similarly, we can derive the following expression for $P_j$:

$$P_j = \left(\frac{T}{j - 1}\right)\left(\frac{m}{D - 1}\right) \times$$

$$\sum_{a=0}^{\min(D, j - 1)} \left(\frac{m - 1}{D}\right) \sum_{b=0}^{D - m} \left(\frac{m - 1}{b}\right) \left(-1\right)^{a + b} \frac{1 - \exp(-\Lambda_i(t))}{R}$$

where we have substituted

$$R = T - j + a + \alpha(D - m + b + 1)$$

From these three expressions we can calculate the score probability, $P(j; T, D, t_i) = P_1 + P_2 + P_3$, from the cumulative hazard function, $\Lambda_i(t)$. To find the cumulative hazard function, $\Lambda_i(t)$, for a distribution function, $F(t)$, we note that it can be calculated as the negative logarithm of the survivor function (Luce, 1991), i.e.:

$$\Lambda_i(t) = -\log(1 - F(t))$$

(Appendices continue)
Thus, all distribution functions including all psychometric functions known to us can be inserted into TVA using the above derivations. Note that in the case of no distractor and only a single target element, the distribution function \( F(t) \) is the psychometric function.

Finally, let us note that our assumption of an in-homogenous Poisson process for visual encoding; rather than a homogenous one as was assumed in (Bundesen, 1990); does not necessarily conflict with the idea of a total processing capacity \( C \); if one assumes that this is no longer constant but rather varying in time. In this way we can use the same formulas for dividing processing resources between elements as used in (Bundesen, 1990). Also we note that the processing rate, \( v \), which was previously a constant, is now a time-varying function \( \lambda(t) \) that corresponds to the hazard of encoding an element.

Appendix B

Formulas for the Psychometric Functions

A psychometric function \( \psi(t; \theta, \gamma, \lambda) \) quantifies the probability of a correct report as a function of some stimulus attribute \( t \), which in our case is exposure duration. It is characterized by a number of parameters that include the parameter set \( \theta \) of the function \( F \) as well as two additional parameters, \( \gamma \) and \( \lambda \), that denote the guessing and lapsing probabilities, respectively. We define the guessing probabilities as the fraction of times an un-informed observer presses (intentionally or accidently) each of the keys included in the response set. The lapsing probability we define as the relative fraction of accidental key presses, averaged over all keys in the response set. The psychometric function \( \psi \), which includes correction for guessing and lapsing, can be written as

\[
\psi(t;\theta,\gamma,\lambda) = F(t;\theta) \cdot (1 - \lambda) + (1 - F(t;\theta)) \cdot \gamma = \gamma + (1 - \gamma - \lambda) \cdot F(t;\theta)
\]

What we shall generally speak of as the psychometric function is the function \( F \), i.e. the psychometric function after correction for guessing and lapsing (Treutwein & Strasburger, 1999; Wichmann & Hill, 2001). In the following we will describe six psychometric functions, which, for various reasons, are plausible candidates for describing letter identification as a function of exposure duration.

The exponential distribution has the parameter set \( \theta = \{\nu, \mu\} \), where \( \nu > 0 \) is the rate and \( \mu > 0 \) is the temporal offset of the Poisson process, and the distribution is defined by:

\[
F(t;\theta) = 1 - e^{-\nu(t-\mu)} \text{ for } t \geq \mu \text{ and } F(t;\theta) = 0 \text{ for } t < \mu.
\]

The Weibull distribution has the parameter set \( \theta = \{\mu, \sigma, k\} \), where \( \mu, \sigma, k > 0 \). It reduces to the exponential distribution when the shape parameter \( k = 1 \). When the parameter \( \mu > 0 \) the Weibull distribution includes an offset. The three-parameter Weibull distribution function is defined as:

\[
F(t;\theta) = 1 - e^{-\left(\frac{t-\mu}{\sigma}\right)^k} \text{ for } t \geq \mu \text{ and } F(t;\theta) = 0 \text{ for } t < \mu.
\]

The gamma distribution has the parameter set \( \theta = \{\mu, \sigma, k\} \), where \( \mu, \sigma, k > 0 \). It reduces to the exponential distribution when \( k = 1 \). Noting that \( \Gamma \) is the complete gamma function and \( \gamma \) is the lower incomplete gamma function,

\[
\Gamma(k) = \int_0^\infty x^{k-1}e^{-x}dx,
\]

\[
\gamma(k,t) = \int_0^t x^{k-1}e^{-x}dx,
\]

the three-parameter gamma distribution function is defined as:

\[
F(t;\theta) = \frac{\gamma(k,t\mu)}{\Gamma(k)} \text{ for } t \geq \mu \text{ and } F(t;\theta) = 0 \text{ for } t < \mu.
\]

The ex-Gaussian distribution has the parameter set \( \theta = \{\mu, \sigma, \tau\} \), where \( \mu \geq 0 \) and \( \sigma, \tau > 0 \). The ex-Gaussian approaches the exponential distribution in the limit, to be exact when \( \mu = \theta \) and \( \sigma \rightarrow 0 \). Noting that \( \Phi \) is the Gaussian distribution function, the ex-Gaussian distribution function is defined as:

\[
F(t;\theta) = \Phi\left(\frac{t - \mu}{\sigma}\right) - \Phi\left(\frac{t - \mu - \sigma^2/\tau}{\sigma}\right) \cdot \exp\left(-\frac{\tau}{\tau} - \frac{\mu}{\tau} + \frac{\sigma^2}{2\tau}\right).
\]

The log-logistic distribution has the parameter set \( \theta = \{\mu, \sigma\} \), where \( \mu, \sigma > 0 \). Noting that the parameter \( \sigma \) determines the steepness, and \( \mu \) is the median survival time, the log-logistic distribution function is defined as:

\[
F(t;\theta) = \frac{1}{1 + \left(\frac{t}{\mu}\right)^{-\sigma}}
\]

The squared-logistic has the parameter set \( \theta = \{V, \mu, \sigma\} \), where \( V, \mu, \sigma > 0 \). We define the squared-logistic distribution function as:

(Appendices continue)
randomized. Furthermore, for each trial the phase content of the mask should be close to the average power spectrum of the stimulus images used. To generate a binary mask that should have a power spectrum that is close to the average power spectrum of the stimulus images used, we need to keep real.

We note that entries in $\Theta$ are chosen randomly for each trial and subject to the constraint that $F$ should be a Hermitian matrix. The entries for $\Theta$ are drawn in conjugated pairs with zero real and random imaginary part. Also note that for the latter equation the DC-frequency and the half sampling frequency $f/2$ were not phase-scrambled, because both frequency components should be kept real.

The discrete inverse Fourier transform of $F^\mu$ can be written as:

$$f^\mu(x_1, x_2) = \sum_{w_1=-\pi}^{\pi} \sum_{w_2=-\pi}^{\pi} F^\mu(w_1, w_2) e^{i w_1 x_1} e^{i w_2 x_2}$$

We now define the rounded average number of pixels $v$ in the $N$ stimulus images that takes on the value 1 rather than 0 as

$$v = \text{round} \left( \frac{1}{N} \sum_{n=1}^{N} \sum_{i=1}^{X_1} \sum_{j=1}^{X_2} f^\mu(x_i, x_j) \right)$$

We now perform a threshold operation so that the $v$ elements in $f^\mu$ that have the largest values are set to one in $M$ and the rest of the elements are set to zero; that is, if we define as $S_v$ the set of elements in $f^\mu$ that have the $v$ largest values, then the random Fourier mask $M$ is found as:

$$M(x_1, x_2) = \begin{cases} 1 & \text{if } f^\mu(x_1, x_2) \in S_v \\ 0 & \text{else} \end{cases}$$

We now explain the method we used for generating a new random mask $M$ for each trial. The idea behind the method is to generate a binary mask that should have a power spectrum that is close to the average power spectrum of the stimulus images used. Furthermore, for each trial the phase content of the mask should be randomized.

Our starting point is that we have $N$ binary stimulus images of size $X_1$ times $X_2$, the $n$'th image $f_n(x_1, x_2)$ has pixel coordinates $x_1$ and $x_2$. Noting that $w_1$ and $w_2$ are the pixel coordinates in frequency space, the discrete Fourier transform of the $n$'th stimulus image can be written as:

$$F_n(w_1, w_2) = \sum_{x_1=1}^{X_1} \sum_{x_2=1}^{X_2} f_n(x_1, x_2) e^{-i \omega_1 x_1} e^{-i \omega_2 x_2}$$

The average Fourier transform $F$ of the $N$ stimulus images can be calculated as:

$$\hat{F}(w_1, w_2) = \frac{1}{N} \sum_{n=1}^{N} F_n(w_1, w_2)$$

Let $\Theta$ be a matrix that has size $X_1$ times $X_2$. We now define the randomly phase-shifted average Fourier transform as:

$$F^\mu(w_1, w_2) = |\hat{F}(w_1, w_2)| \begin{cases} 1 & \text{if } w_1 = 0 \wedge w_2 = 0 \quad \text{(DC)} \\ 1 & \text{if } w_1 = \frac{1}{2X_1} \wedge w_2 = \frac{1}{2X_2} \quad \text{(N)} \\ \exp(\Theta) & \text{else} \end{cases}$$

The average Fourier transform $F$ is given by:

$$F(t; \theta) = 1 - e^{-\left(1 + e^t \left(\frac{\theta}{2\pi}\right)^2\right)^{-1}}$$

We named the distribution the squared-logistic because, the shape of the mean cumulative hazard function in the interval between 0 and $t$ as a function of time has the shape of a logistic distribution function squared. This can be seen by dividing the negative exponent (the cumulative hazard function) of the distribution by the size $t$ of the temporal interval. Note that the parameter $V$ scales the hazard rate of the squared-logistic. Furthermore, we can note that it is straightforward to derive the probability density function of the squared-logistic, if needed.

### Appendix C

#### Generating a Random Fourier Mask

We now perform a threshold operation so that the $v$ elements in $f^\mu$ that have the largest values are set to one in $M$ and the rest of the elements are set to zero; that is, if we define as $S_v$ the set of elements in $f^\mu$ that have the $v$ largest values, then the random Fourier mask $M$ is found as:

$$M(x_1, x_2) = \begin{cases} 1 & \text{if } f^\mu(x_1, x_2) \in S_v \\ 0 & \text{else} \end{cases}$$

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